Control of chaos by occasional bang-bang

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The stabilization of a periodic saddle orbit (the *target orbit*) in a strange attractor is usually achieved by the application of a sequence of parameter perturbations designed to place the system state in the stable manifold of the target, whereupon it evolves unperturbed to the target orbit. Controls of this type originated with the method of Ott, Grebogi, and Yorke (OGY), and usually require a continuously variable parameter for map based control. Bang-bang control is a method whereby control is achieved by the application of a fixed, or several different fixed perturbations, rather than a continuous range of perturbations, and generally requires a flexible scheduling. If we have available only fixed perturbation levels, and can apply control only at regular intervals and for fixed durations, standard OGY will not work. We demonstrate a method that will control maps and continuous systems at a surface of section, albeit imprecisely, with a single fixed perturbation of fixed duration. We call this occasional bang bang.

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Much work has been done on so-called bang-bang control in the classical control theory literature [1–3], but very little has been written from the viewpoint of chaos control. Starrett and Tagg [4] showed that a parametrically forced chaotic pendulum could be controlled by the application of a fixed perturbation for an amount of time proportional to the deviation of the current state from the target orbit. This type of bang-bang control is known as time proportioned perturbations (TPP), and uses a controller whose on-time is variable. Epureanu and Dowell [5] considered a one-dimensional OGY (Ott, Grebogi, and Yarke) problem in the course of investigation of higher-dimensional OGY control and found that a bang-bang solution was optimal for that case.

We propose a controller that is effective in controlling an orbit nearby a target orbit when the controller can take on only one or two perturbed values and only one value of on-time, and demonstrate it on a model of a parametrically forced pendulum. As with OGY [7] control, the control feedback is derived solely from information taken from a map or surface of section map and the control on-time is equal to a single iterate of the map, that is, the control remains on for a full forcing cycle. Throughout this paper, we assume we are dealing with a two-dimensional map, or a periodically forced three-dimensional system with a two-dimensional Poincaré map, whose dynamics near saddle orbits are nearly linear.

Suppose we have a chaotic map $\mathbf{x}_{i+1}=f(\mathbf{x}_i,\rho)$, which has a saddle orbit $\overline{\mathbf{x}}$ we wish to stabilize, and a control parameter ρ that can take on only two values, the nominal value ρ_0, and a perturbation ρ_1. With only two values of perturbation, it is impossible to guide a nearby orbit $\hat{\mathbf{x}}$ to the target orbit $\overline{\mathbf{x}}$, or even to linear stable manifold $\mathbf{e}_s$ of the target orbit, as with OGY control. We settle instead for stabilizing another orbit, or set of orbits, in a small region near the target orbit. Control is achieved by applying the perturbation irregularly, scheduled so as to guarantee that the system state will be alternately driven towards the stable manifold of the unperturbed target orbit and allowed to move away a little before control is applied again.

I. OBB CONTROL OF A TOY SYSTEM

The method of occasional bang-bang control can be simply demonstrated on a one-dimensional expanding map. Let $f:\mathbb{R}\to\mathbb{R}$ with

$$f(x):x_{n+1}=\lambda x_n,\quad \lambda\in(1,2)$$

a linear expanding map. Then the action of the map on almost all initial conditions $x_0\in[-1,1]$ maps x_m out of $[-1,1]$ for some values of m. If we replace f with

$$F(x)=\begin{cases} x_{i+1}=\lambda(x_i+1)-1, & x_i\in[-1,-1/2]\\ x_{i+1}=\lambda x_i, & x_i\in(-1/2,1/2)\\ x_{i+1}=\lambda(x_i-1)+1, & x_i\in[1/2,1]\\ x_{i+1}=\lambda x_i, & x_i\in(-\infty,1),(1,\infty)\end{cases}\tag{1}$$

then we have replaced an uncontrolled system $f(x)$ with a controlled system $F(x)$, because for any initial condition $x_0\in[-1,1]$, the orbit $F^{(j)}(x_0)$ will remain in $[-1,1]$ for all j. The control is off for $x_j\in(-1/2,1/2)$ and is turned on for $x_j\in[-1,-1/2]$ or $[1/2,1]$ and remains on until x_j is again in $(-1/2,1/2)$. $F(x)$ is a member of a family of maps known as shift maps or Bernoulli maps, and f and $F(x)$ are shown in Fig. 1.

We can use the same control strategy for a two-dimensional toy system. Consider the map $\mathbf{x}_{i+1}=A\mathbf{x}_i$, where A is a 2×2 matrix with distinct unit eigenvectors $\mathbf{e}_s,\mathbf{e}_u$ and associated eigenvalues $0<\lambda_s<1<\lambda_u<2$. Let $\check{\mathbf{x}}=\check{x}\mathbf{e}_u+\check{y}\mathbf{e}_s$ be the state vector written in A's eigenvector basis. Then for the coefficients $\check{x},\check{y}$, the map

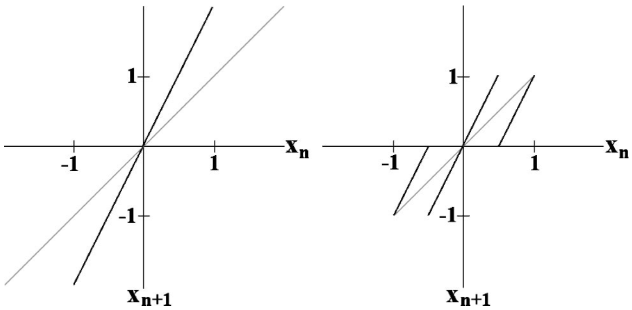

FIG. 1. The map on the left, f is uncontrolled, as any initial condition x_0 in $[-1,1]$ will escape exponentially fast, while the variation on the right, $F(x)$ is controlled, in that any initial condition x_0 in $[-1,1]$ will remain in $[-1,1]$ for all time. Units are nondimensional.

$$G(x,y)=\begin{cases} \breve{x}_{i+1}=\lambda_u(\breve{x}_i+1)-1, & \breve{x}_i\in[-1,-1/2] \\ \breve{y}_{i+1}=\lambda_s(\breve{y}_i+1)-1, & \breve{x}_i\in[-1,-1/2] \\ \breve{x}_{i+1}=\lambda_u\breve{x}_i, & \breve{x}_i\in(-1/2,1/2) \\ \breve{y}_{i+1}=\lambda_s\breve{y}_i, & \breve{x}_i\in(-1/2,1/2) \\ \breve{x}_{i+1}=\lambda_u(\breve{x}_i-1)+1, & \breve{x}_i\in[1/2,1] \\ \breve{y}_{i+1}=\lambda_s(\breve{y}_i-1)+1, & \breve{x}_i\in[1/2,1] \\ \breve{x}_{i+1}=\lambda_u\breve{x}_i, & \breve{x}_i\in(-\infty,1),(1,\infty) \\ \breve{y}_{i+1}=\lambda_s\breve{y}_i, & \breve{x}_i\in(-\infty,1),(1,\infty) \end{cases} \tag{2}$$

acts as a controlled system, with the dynamics along the $\mathbf{e}_u$ direction mimicking the action of the Bernoulli map of Eq. (1). Note that the dynamics in the $\mathbf{e}_s$ direction are independent of the $\breve{y}$ coefficient. A graph of an orbit of this two-dimensional toy system with $A=\left[\begin{smallmatrix}0.6 & -0.1\\ 0.21 & 1.6\end{smallmatrix}\right]$ is shown in Fig. 2.

II. OBB FOR SIMPLE SADDLES

Suppose we have a two-dimensional chaotic map f with state variable $\hat{\mathbf{x}}$ whose linearization near a saddle orbit (the target orbit) $\overline{\mathbf{x}}$ is $\mathbf{x}_{i+1}=A\mathbf{x}_i$, where A is a 2×2 matrix with eigenvalues $0<\lambda_s<1<\lambda_u<2$, associated unit eigenvectors $\mathbf{e}_s,\mathbf{e}_u$, and $\mathbf{x}=\hat{\mathbf{x}}-\overline{\mathbf{x}}$. Further suppose the system has a parameter ρ whose nominal value is set to ρ_0, and we allow for one level of perturbation $\delta\rho=\rho_1-\rho_0$, where ρ_1 is the perturbed parameter value. The change in the location of the target orbit with ρ is $\mathbf{g}=\partial\overline{\mathbf{x}}/\partial\rho$, and we assume $\mathbf{g}$ does not lie along the stable manifold $\mathbf{e}_s$ (otherwise the system would be uncontrollable). The dynamics under the perturbation $\delta\rho_i$ will be $\mathbf{x}_{i+1}-\delta\rho_i\mathbf{g}=A(\mathbf{x}_i-\delta\rho_i\mathbf{g})$.

Denote by $\tilde{\mathbf{e}}_s,\tilde{\mathbf{e}}_u,\tilde{\mathbf{x}}$ the perturbed stable and unstable unit eigenvectors, and the perturbed target orbit, respectively. When the system enters a controllable region bounded by lines along the eigenvectors $\tilde{\mathbf{e}}_s,\tilde{\mathbf{e}}_u,\mathbf{e}_s,\mathbf{e}_u$, the controller decides to turn on or off based on whether the system state is nearer the perturbed $\tilde{\mathbf{e}}_s$ or the unperturbed $\mathbf{e}_s$ stable manifolds. Write $\delta\mathbf{g}$ in a basis of the eigenvectors of A, i.e., $\delta\mathbf{g}=g_s\mathbf{e}_s+g_u\mathbf{e}_u$. Then a line d from $g_s\mathbf{e}_s+(g_u/2)\mathbf{e}_u$ to $(g_u/2)\mathbf{e}_u$ divides the parallelogram bounded by $\tilde{\mathbf{e}}_s,\tilde{\mathbf{e}}_u,\mathbf{e}_s,\mathbf{e}_u$ into two equal parts. The region bounded in the unstable direction by d and $\mathbf{e}_s$ we call the *outbox*, and one bounded by d and $\tilde{\mathbf{e}}_s$ we call the *inbox*. The names are chosen because points in the inbox will be driven *in* towards $\mathbf{e}_s$, and those in the outbox will be driven *out*, or away, from $\mathbf{e}_s$. Figure 3 illustrates the geometry of the control. If the system state is in the inbox, the control will be on and the system will be driven towards the unperturbed stable manifold for as many iterates as it takes for the system to enter the outbox. Once in the outbox, it will evolve under the unperturbed dynamics until it enters the inbox, and the cycle will begin anew.

III. OBB FOR FLIP SADDLES

While linear saddles of true maps in two dimensions can have eigenvalues of mixed sign, Poincaré maps extracted

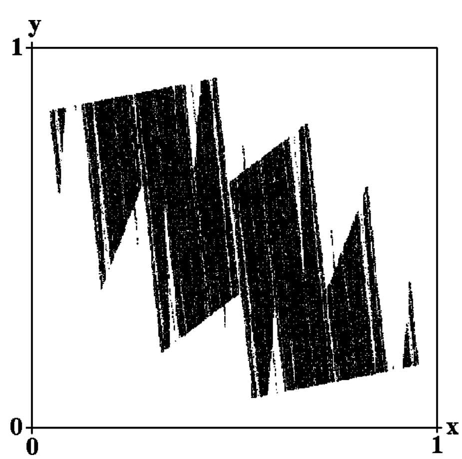

FIG. 2. Orbit of a toy system with eigenvalues $\lambda_u\approx1.62$, $\lambda_s\approx0.58$, and eigenvectors $\mathbf{e}_u\approx(0.1,-1.0)$, $\mathbf{e}_s\approx(-0.98,-0.2)$. The units are nondimensional.

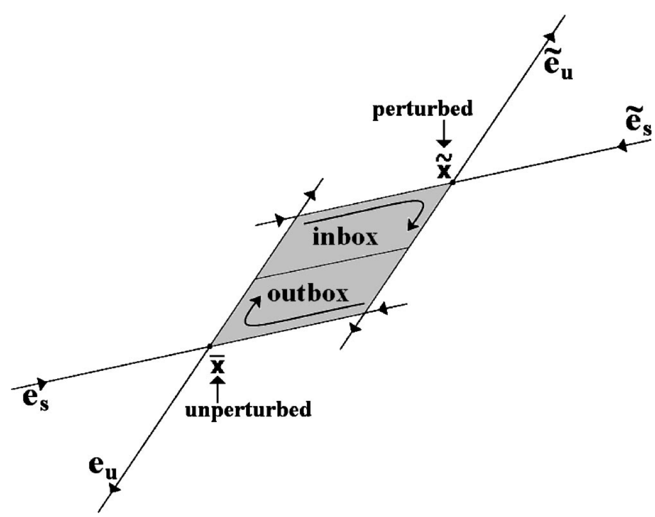

FIG. 3. The perturbed and unperturbed stable and unstable eigenvectors of the local linear map enclose a controllable region (in gray). This region is divided into two subregions determined by the dynamics, which are in effect for system states in that region. States in the *inbox* evolve under the perturbed dynamics, and those in the *outbox* evolve under the unperturbed dynamics.

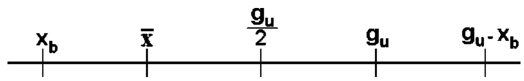

FIG. 4. The one-dimensional control setup associated with OBB for flip saddles with one level of perturbation. The target orbit is labeled $\bar{\mathbf{x}}$ and the perturbed target orbit is labeled g_u. Units are nondimensional.

from continuous three-dimensional systems support only simple and flip saddles, whose eigenvalues are both positive or both negative, respectively. Should the target orbit be a flip saddle, then the control is complicated by the flipping of the orbit on either side of the manifold.

As with a simple saddle, the size of the controllable region is determined by the dynamics in the unstable direction, so we consider the action of a one-dimensional map along $\mathbf{e}_u$. Let $\mathbf{x}_i$ lie in $\mathbf{e}_u$. Then the map $\mathbf{x}_{i+1}=A\mathbf{x}_i$ reduces to $\breve{x}_{i+1}=\lambda_u\breve{x}_i$, where, as before, $\breve{x}$ is the component of $\mathbf{x}$ along the $\mathbf{e}_u$ direction. The OBB controlled version of this map is

$$\breve{x}_{i+1}=\lambda_u\breve{x}_i, \quad \breve{x}_i<x_b$$

$$\breve{x}_{i+1}=\lambda_u\breve{x}_i, \quad \breve{x}_i\in\left[x_b,\frac{g_u}{2}\right]$$

$$\breve{x}_{i+1}=\lambda_u(\breve{x}_i-g_u)+g_u, \quad \breve{x}_i\in\left(\frac{g_u}{2},g_u-x_b\right] \tag{3}$$

$$\breve{x}_{i+1}=\lambda_u\breve{x}_i, \quad \breve{x}_i>g_u-x_b,$$

where x_b, g_u-x_b are the as yet undetermined bounds of the controllable region. Figure 4 shows the geometry of the control along $\mathbf{e}_u$. As before, the dynamics in the $\mathbf{e}_s$ direction do not enter into control decisions.

OBB loses control when $\breve{x}_j<x_b$ or $\breve{x}_j>g_u-x_b$. Since the controlled system is symmetric about $g_u/2$, a period two orbit between x_b and g_u-x_b separates the controllable region from the uncontrollable region. Setting $x_b=\breve{x}_i$, we have $\breve{x}_{i+2}=\lambda_u(\lambda_u\breve{x}_i-g_u)+g_u$, so

$$x_b=g_u\frac{1}{1+\lambda_u}, \quad g_u-x_b=g_u\frac{\lambda_u}{1+\lambda_u}.$$

It is possible with this type of control for points near $g_u/2$ to map out of $[g_u(1/1+\lambda_u),g_u(\lambda_u/1+\lambda_u)]$. In order to ensure a continuous controllable region, we require that $g_u/2$ be controllable, that is, we require $\lambda_u(g_u/2)\geqslant g_u(1/1+\lambda_u)$. Solving this for λ_u, we find the system has a continuous controllable region for $-2\leqslant\lambda_u\leqslant-1$. Thus, OBB control with one fixed level of perturbation is effective for flip saddles whose unstable eigenvalue satisfies $-2\leqslant\lambda_u\leqslant-1$, and over a region bounded in the unstable direction by lines parallel to $\mathbf{e}_s$ and through $g_u(1/1+\lambda_u)\mathbf{e}_u$ and $g_u(\lambda_u/1+\lambda_u)\mathbf{e}_u$.

If we have available two equal and opposite perturbations, a similar calculation shows that OBB is effective for flip saddles whose unstable eigenvalue satisfies $-(1+\sqrt{17}/2)\leqslant\lambda_u\leqslant-1$, and over a region bounded in the unstable direction by lines parallel to $\mathbf{e}_s$ and through $-g_u(\lambda_u-1/\lambda_u+1)\mathbf{e}_u$ and $g_u(\lambda_u-1/\lambda_u+1)\mathbf{e}_u$.

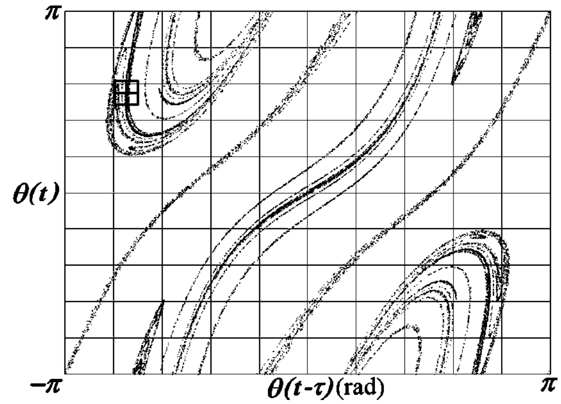

FIG. 5. The reconstructed attractor for the over-the-top mode of the parametrically (sinusoidal vertical) forced pendulum, where θ is the angular position and τ is the delay time.. The control box is in the upper left of the fourth quadrant, and is nearly square since the stable and unstable manifolds are nearly orthogonal and aligned with the coordinates.

The boundaries of the controllable region in the stable direction are not as easily determined. An intelligent choice may depend on the linearity of the stable manifold, the location of the saddle in the attractor, the relative size of the stable and unstable eigenvalues, among other considerations (see Ref. [6] for arguments about controllable regions and linearity). We have found that a rhombus with sides parallel to $\mathbf{e}_u, \mathbf{e}_s$ and centered on the target orbit works satisfactorily.

OBB control for flip saddles with one level of perturbation, therefore, consists of waiting until the system enters the controllable region, then applying perturbations according to

$$\text{perturbation off} \quad \breve{x}\in\left[g_u(1/1+\lambda_u),\frac{g_u}{2}\right],$$

$$\text{perturbation on} \quad \breve{x}\in(g_u/2,g_u(\lambda_u/1+\lambda_u)]. \tag{4}$$

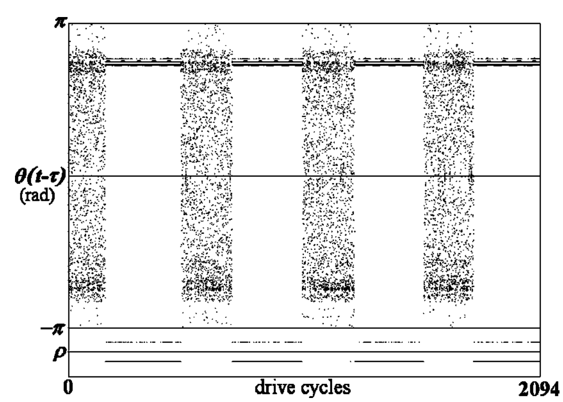

FIG. 6. A time series of the controlled orbit of the pendulum, with the delayed coordinate $\theta(t-\tau)$, where τ is the delay time, on the vertical axis. The three levels of control perturbation are shown magnified by a factor of five in the bottom box, with the center being the nominal control value $\rho=0.23$. The control was turned on and off four times over the course of 2094 drive cycles.

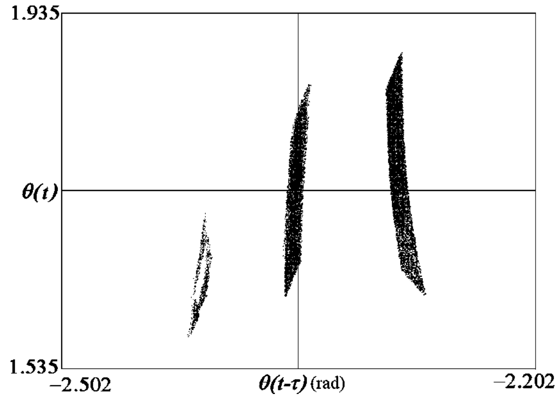

FIG. 7. A closeup of the Poincaré section of the controlled orbit of the pendulum. The bounding box is the control box of Fig. 5.

Likewise, for control with two equal and opposite levels of perturbation $\delta\rho_-$ and $\delta\rho_+$, we apply

$$\begin{aligned} &\text{perturbation } \delta\rho_- \quad \check{x}\in[-g_u(\lambda_u-1/\lambda_u+1),-g_u/2],\\ &\text{perturbation off} \quad \check{x}\in[-g_u/2,g_u/2], \qquad (5)\\ &\text{perturbation } \delta\rho_+ \quad \check{x}\in[g_u/2,g_u(\lambda_u-1/\lambda_u+1)]. \end{aligned}$$

IV. CONTROL OF A PENDULUM BY OBB

We controlled a model of a vertically forced pendulum with damping by the two-perturbation-method of Eq. 5. The nondimensional equation of motion we use is

$$\ddot{\theta}=\rho\dot{\theta}+\sin\theta[1-\alpha\cos(\omega t)],$$

where ρ is velocity dependent damping, θ is the angle measured counterclockwise from the straight down position, α is the amplitude of the forcing, and ω is the forcing frequency. In our study, $0.21\leqslant\rho\leqslant0.25$ with $\delta\rho_{off}=0.23$, $\alpha=1.2$, and $\omega=1.5$. The simulation uses time-delay coordinates with the delay time τ equal to 12/25 of the forcing cycle. Figure 5 shows the attractor of the pendulum in its uncontrolled state.

Figure 6 shows a time series control graph of the pendulum. The modified orbit is centered on the target orbit, which lies in the center of the middle region of the controlled orbit.

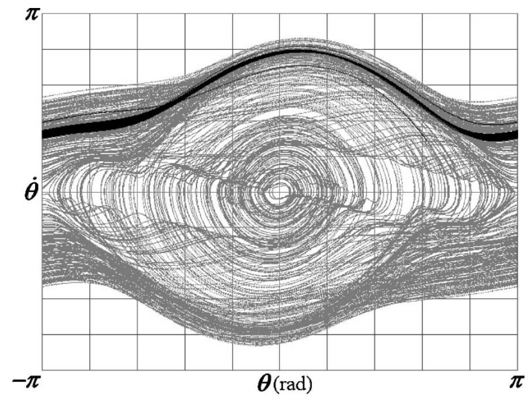

FIG. 8. The phase portrait of the controlled orbit (black) and the uncontrolled orbit (gray) of the pendulum in natural coordinates $(\theta,\dot{\theta})$.

Figure 7 shows a closeup of the modified orbit created by the control. The target orbit is at the center of the crosshairs, while the actual controlled orbit consists of three pieces: the center piece, where iterates are in the outbox, and the left and right pieces, which are iterates in one of the two inboxes. The target orbit in this example has a stable eigenvalue λ_s whose magnitude is near 0.2, so the contraction of the map is relatively strong. This accounts for the clustering of states near the perturbed target orbits. Figure 8 shows both the controlled orbit (black) and the uncontrolled orbit (gray) in the phase plane in natural coordinates.

V. SUMMARY

In summary, the method of occasional bang-bang is capable of stabilizing a chaotic system represented by a two-dimensional map nearby any one of its periodic orbits by occasional fixed perturbations for a fixed length of time, provided the magnitude of the unstable eigenvalue lies between limits that depend on the saddle type. We successfully demonstrated the method on a period-one flip orbit of a model of a vertically forced chaotic pendulum. Because of the simplicity of its control rule, it may be useful for fast systems, for example, diode oscillators or lasers, without the need for computer calculation, since a comparator can decide to turn the control on or off based on a predetermined voltage level.

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